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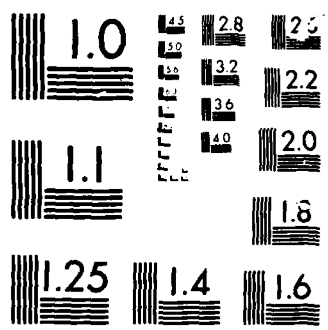

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A FREE STREAMLINE MODEL FOR A RISING BUBBLE

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A FREE STREAMLINE MODEL FOR A RISING BUBBLE

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ABSTRACT

The motion of a two-dimensional bubble rising at a constant velocity in an unbounded fluid is solved by series truncation. It is assumed there is a wake of stagnant liquid extending to infinity below the bubble. Both the effects of gravity g and surface tension T are taken into account. It is shown that the problem is characterized by a continuum of solutions for $T = 0$ and by a discrete set of solutions when $T > 0$. In addition a unique solution is obtained in the limit as the surface tension approaches zero. The corresponding profile of the bubble is found to be in good agreement with experimental data.

AMS (MOS) Subject Classification: 76B10

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SIGNIFICANCE AND EXPLANATION

Recent numerical computations on nonlinear free surface flow problems (Vanden-Broeck[3,4,5]) have uncovered an unexpected effect of surface tension. It has been found that some problems are characterized by a continuum of solutions when surface tension is neglected and by a discrete set of solutions when surface tension is taken into account.

In the present paper we describe another example of such flows. We consider the motion of a two-dimensional bubble rising at a constant velocity in an unbounded fluid. It is assumed there is a wake of stagnant liquid extending to infinity below the bubble. Both the effects of gravity and surface tension are taken into account. It is found that this problem is also characterized by a continuum of solutions when surface tension is neglected and by a discrete set of solutions when surface tension is taken into account. Moreover it is shown that a unique solution is obtained in the limit as the surface tension tends to zero. The corresponding profile of the bubble is found to be in good agreement with experimental data.

A FREE STREAMLINE MODEL FOR A RISING BUBBLE

Jean-Marc Vanden-Broeck

1. Introduction

We consider a bubble rising at a constant velocity U in an unbounded inviscid fluid. We take a frame of reference moving with the bubble and we assume there is a wake of stagnant liquid extending to infinity below the bubble (see Fig. 1).

As we shall see the problem is characterized by the Froude number

$$F = U/(gD)^{1/2} \tag{1}$$

and the Weber number

$$\alpha = \frac{\rho U^2 D}{T} . \tag{2}$$

Here g is the acceleration of gravity, T the surface tension, ρ the density of the liquid and D the width of the bubble (i.e. the distance between the separation points S and S').

In this paper we present numerical evidence that for $\alpha = \infty$ (i.e. $T = 0$) there is a solution for each value of $0 < F < F_c$ where $F_c \sim 0.9$. However, for each value of $\alpha \neq \infty$ there exists only a countably infinite number of solutions. Each of these solutions corresponds to a different value of F. As α tends to infinity, all these solutions approach a unique limiting solution characterized by $F = F^* \sim 0.51$. The corresponding profile of the bubble is found to be in good agreement with Collins'[1] experimental data.

The present results are similar to those obtained by McLean and Saffman[2] and Vanden-Broeck[3] for the viscous flow in a Hele-Shaw cell and by Vanden-Broeck[4,5,6] for the flow past a bubble in a tube. All three problems are

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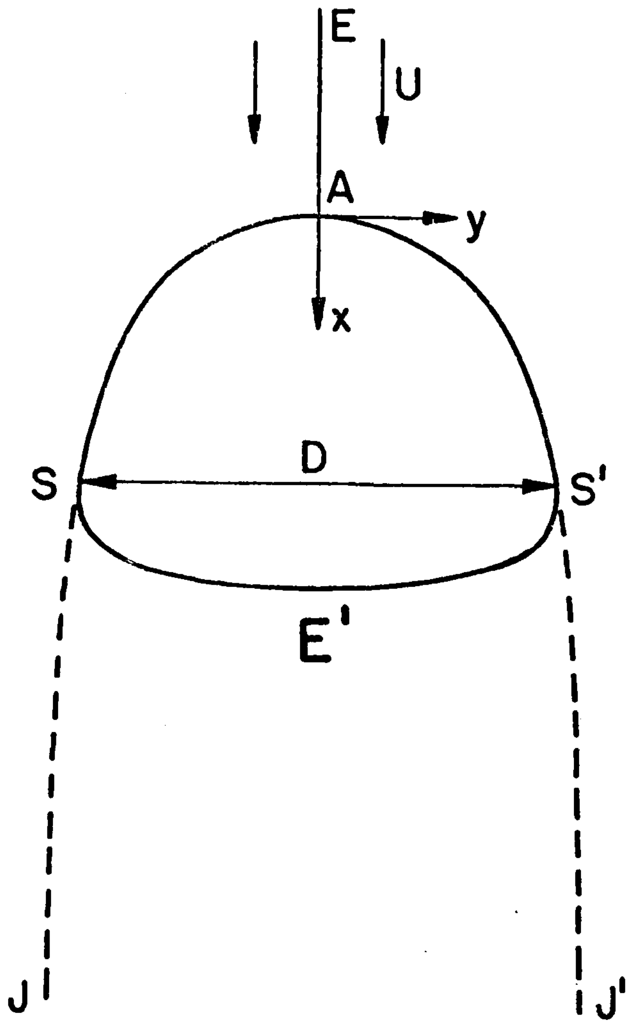

Figure 1: Sketch of the flow and of the coordinates. The solid line corresponds to the bubble profile and the broken lines to the wake profile.

characterized by a continuum of solutions for $T = 0$ and a discrete set of solutions for $T > 0$.

The problem is formulated in Section 2. The numerical procedure is presented in Section 3 and the results are discussed in Section 4 and 5.

2. Formulation

Let us consider the steady two-dimensional potential flow of an inviscid incompressible fluid past a bubble (see Fig. 1). The constant pressure in the bubble is denoted by P_b. We introduce Cartesian coordinates with the origin at the top of the bubble and we assume that the bubble is symmetric about the x-axis. Gravity acts in the positive x-direction.

We approximate the wake behind the bubble by a free streamline model. Therefore the velocity is equal to U on the surfaces SJ and $S'J'$ of the wake. Inside the wake the fluid is at rest and the pressure is hydrostatic.

We define dimensionless variables by taking U as the unit velocity and D as the unit length. We introduce the potential function ϕb and the stream function ψb. The constant b is chosen such that $\phi = 1$ at the separation points S and S'. Without loss of generality we choose $\phi = 0$ at $x = y = 0$ and $\psi = 0$ on the surface of the bubble and of the wake.

We denote the complex velocity by $u - iv$ and we define the function $\tau - i\theta$ by the relation

$$u - iv = e^{\tau - i\theta} . \tag{3}$$

We shall seek $\tau - i\theta$ as an analytic function of $f = \phi + i\psi$ in the half plane $\psi < 0$. The complex potential plane is sketched in Figure 2. At infinity we require the velocity to be unity in the x-direction so that $\tau - i\theta$ vanishes at infinity in view of (3).

On the surface of the bubble the Bernoulli equation and the pressure jump due to surface tension yield

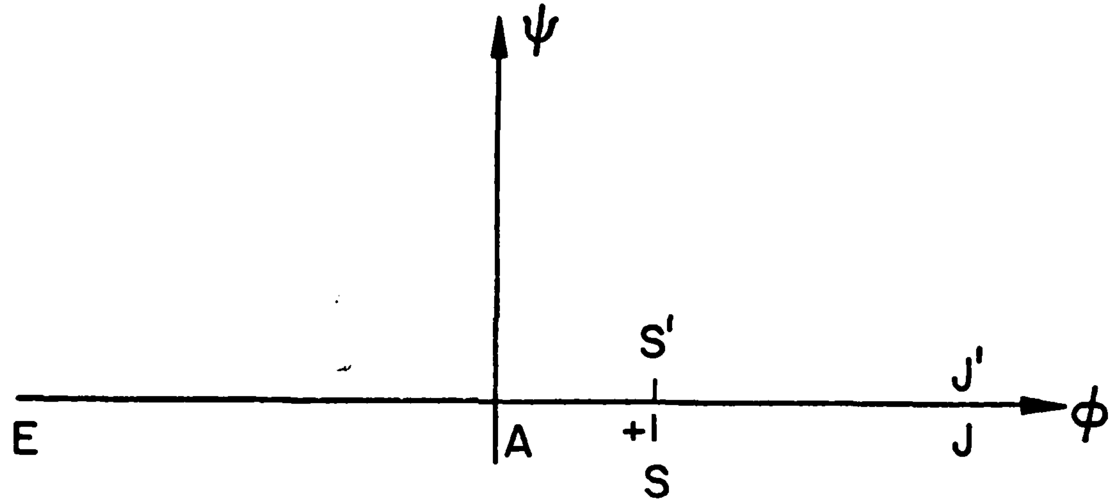

Figure 2: The image of the flow in the complex potential plane $f = \varphi + i\psi$.

$$\frac{1}{2} q^2 - gx + \frac{T}{\rho} K = B + \frac{p_b}{\rho} \quad \text{on} \quad SAS' \ , \tag{4}$$

$$\frac{p_w}{\rho} - \frac{p_b}{\rho} = \frac{T}{\rho} K \quad \text{on} \quad SES' \ . \tag{5}$$

Here q is the flow speed, K the curvature of the bubble surface counted positive when the bubble is on the concave side of the surface, B the Bernoulli constant and p_w the hydrostatic pressure in the wake. In dimensionless variables (4) becomes

$$e^{2\tau} - \frac{2}{F^2} x - \frac{2}{\alpha} e^{\tau} \frac{\partial \theta}{\partial \phi} = \frac{2B}{U^2} , \ \psi = 0, \ 0 < \phi < 1 \ . \tag{6}$$

Here F and α are the Froude and Weber number defined by (1) and (2) respectively.

On the surface SJ and $S'J'$ of the wake, the velocity is equal to U. In dimensionless variables this yields

$$\tau = 0, \quad \psi = 0, \quad \phi > 1 \ . \tag{7}$$

It is convenient to eliminate y, p_b and B from (6) by differentiating (6) with respect to ϕ. Using the relation

$$\frac{\partial x}{\partial \phi} + i \frac{\partial y}{\partial \phi} = \frac{1}{u - iv} = e^{-\tau + i\theta} \tag{8}$$

we obtain

$$e^{2\tau} \frac{\partial \tau}{\partial \phi} + \frac{1}{F^2} e^{-\tau} \cos \theta - \frac{1}{\alpha} \frac{\partial}{\partial \phi} \left(e^{\tau} \frac{\partial \theta}{\partial \phi}\right) = 0, \ \psi = 0, \ 0 < \phi < 1 \ . \tag{9}$$

At the top of the bubble, the velocity is equal to zero and the profile of the free surface is tangent to the x-axis. This yields the conditions

$$\tau = -\infty \quad \text{at} \quad \phi = \psi = 0 \tag{10}$$

$$\theta = 0 \quad \text{at} \quad \phi = \psi = 0 \ . \tag{11}$$

Finally the symmetry of the problem implies

$$\theta = 0, \quad \psi = 0, \quad \phi < 0 \ . \tag{12}$$

This completes the formulation of the problem of determining the function $\tau - i\theta$ and the constant b. For ech value of α and F, $\tau - i\theta$ must be analytic in the half-plane $\psi < 0$ and satisfy the boundary conditions (7), (9)-(12). This system of equations is solved in the next section. Once the function $\tau - i\theta$ is known, the shape of the upper suface SAS' of the bubble is obtained by integrating numerically (8) on $\psi = 0$ from $\phi = 0$ to $\phi = 1$.

The Bernoulli equation and the continuity of the pressure across the surface SJ and S'J' of the wake yield an expression for P_w inside the wake:

$$\frac{P_w}{\rho} = B + gx - \frac{U^2}{2} . \tag{13}$$

Upon substituting (13) into (5) we obtain a simple second order differential equation for the shape of the lower surface SES' of the bubble. Two boundary conditions for this equation are obtained by imposing the continuity of the bubble profile at S and S'. In the particular case $\alpha = \infty$ (i.e. $T = 0$) the lower surface of the bubble is simply a straight line from S to S'.

3. Numerical Procedure

We define the new variable t by the transformation

$$f^{1/2} = (t - \frac{1}{t}) \frac{1}{2i} . \tag{14}$$

This transformation maps the flow domain onto the inside of the unit circle. The problem in the complex t-plane is illustrated in Fig. 3.

We use the procedure derived by Vanden-Broeck[3] to investigate the effect of surface tension on the shape of fingers in a Hele-Shaw cell (see also Vanden-Broeck[5]). Thus we define a modified problem by replacing (11) by

$$\theta = \gamma, \quad \phi = \psi = 0 . \tag{15}$$

Here γ is to be found as part of the solution.

We will solve the modified problem defined by (7), (9), (10), (12) and (15) for all values of F and α. Then we will obtain the solutions of the original problem by selecting among the solutions of the modified problem those for which $\gamma = 0$.

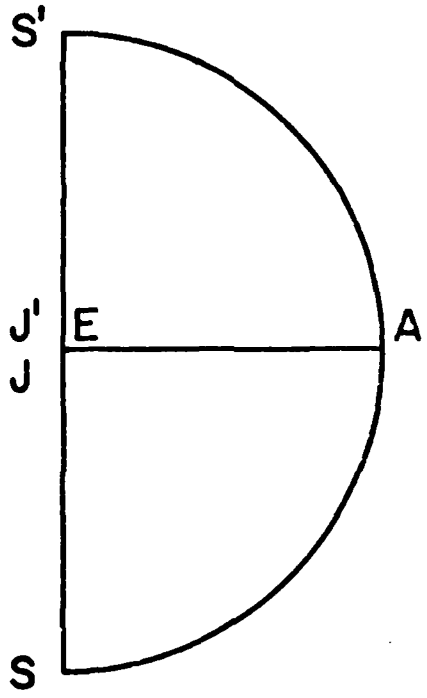

Figure 3: The complex t-plane.

Following Brodetsky[7] we introduce the function $\Omega(t)$ by the relation

$$\tau - i\theta = -\frac{\gamma}{\pi}\log\frac{1+t}{1-t} - \Omega(t) \ . \tag{16}$$

The conditions (7) and (12) show that $\Omega(t)$ can be expressed as a Taylor expansion in odd powers of t. Hence

$$\tau - i\theta = -\frac{\gamma}{\pi}\log\frac{1+t}{1-t} - \sum_{n=1}^{\infty} A_n t^{2n-1} \ . \tag{17}$$

The function (17) satisfies the conditions (7), (10), (12) and (15). The coefficients A_n and b have to be determined to satisfy the condition (9) on the surface SAS' of the bubble. We use the notation $t = re^{i\sigma}$ so that points on SAS' are given by $r = 1$, $-\frac{\pi}{2} < \sigma < \frac{\pi}{2}$. Using (3) and (14) we have

$$\frac{d\tilde{x}(\sigma)}{d\sigma} = b\,\sin 2\sigma e^{-\tilde{\tau}(\sigma)}\cos\tilde{\theta}(\sigma) \ , \tag{18}$$

$$\frac{d\tilde{y}(\sigma)}{d\sigma} = b\,\sin 2\sigma e^{-\tilde{\tau}(\sigma)}\sin\tilde{\theta}(\sigma) \tag{19}$$

where $\tilde{x}(\sigma)$, $\tilde{y}(\sigma)$, $\tilde{\tau}(\sigma)$ and $\tilde{\theta}(\sigma)$ denote the values of x, y, τ and θ on the free surface SA.

We solve the problem approximately by truncating the infinite series in (17) after $N-1$ terms. We find $N-1$ coefficients A_n and the constant b and γ by collocation. Thus we introduce the N mesh points

$$\sigma_I = -\frac{\pi}{2N} I, \qquad I = 1,\dots,N \ . \tag{20}$$

Using (17)-(19) and (20) we evaluate $\tilde{\tau}(\sigma_I)$, $\tilde{\theta}(\sigma_I)$ and $\left(\frac{d\tilde{x}}{d\sigma}\right)_{\sigma=\sigma_I}$ in terms of the coefficients A_n and the constants b and γ. Substituting these expressions into (9) we obtain N algebraic equations for the $N+1$ unknowns A_n, b and γ.

The last equation is obtained by specifying

$$\tilde{y}\left(-\frac{\pi}{2}\right) = -\frac{1}{2} \ . \tag{21}$$

We solve this system by Newton method for given values of F and α. Once this system is solved, we obtain the profile of the bubble by following the procedure described at the end of Section 2.

4. Solutions Without Surface Tension

We used the scheme presented in Section 3 to compute solutions with $\alpha = \infty$ (i.e. $T = 0$). The numerical results were found to be similar to those obtained by Vanden-Broeck[4,6] for the flow past a two-dimensional bubble in a tube. In particular we found

$$\gamma = \frac{\pi}{2} \qquad F < F_c \sim 0.9 , \tag{22}$$

$$\gamma = \frac{\pi}{3} \qquad F = F_c \sim 0.9 , \tag{23}$$

$$\gamma = 0 \qquad F > F_c \sim 0.9 . \tag{24}$$

As $F \to \infty$ the free surfaces collapse to the vertical line $y = 0$ and the flow reduces to a uniform stream. As $F \to 0$, the free surfaces approach the horizontal line $x = 0$ and the problem reduces to the classical Kirchhoff model for cavitating flow past a flat plate.

Relations (22)-(24) show that all the solutions corresponding to $F < F_c$ are solutions of the original problem. The solutions for $F > F_c$ are only solutions of the modified problem. They are characterized by a discontinuity in slope at the apex of the bubble.

In Fig. 4 we present values of the dimensionless velocity q_A/U at the apex A of the bubble vs F. The velocity q_A is equal to zero for $F < F_c$ and is different from zero for $F > F_c$.

Collins[1] performed experiments for two dimensional bubbles rising in a fluid. He observed that the profile of the bubbles were close to circular arcs. Using a method due to Davies and Taylor[8], he derived analytically an approximate relation between the velocity U at which the bubble rises and the radius R of the circular arc, namely

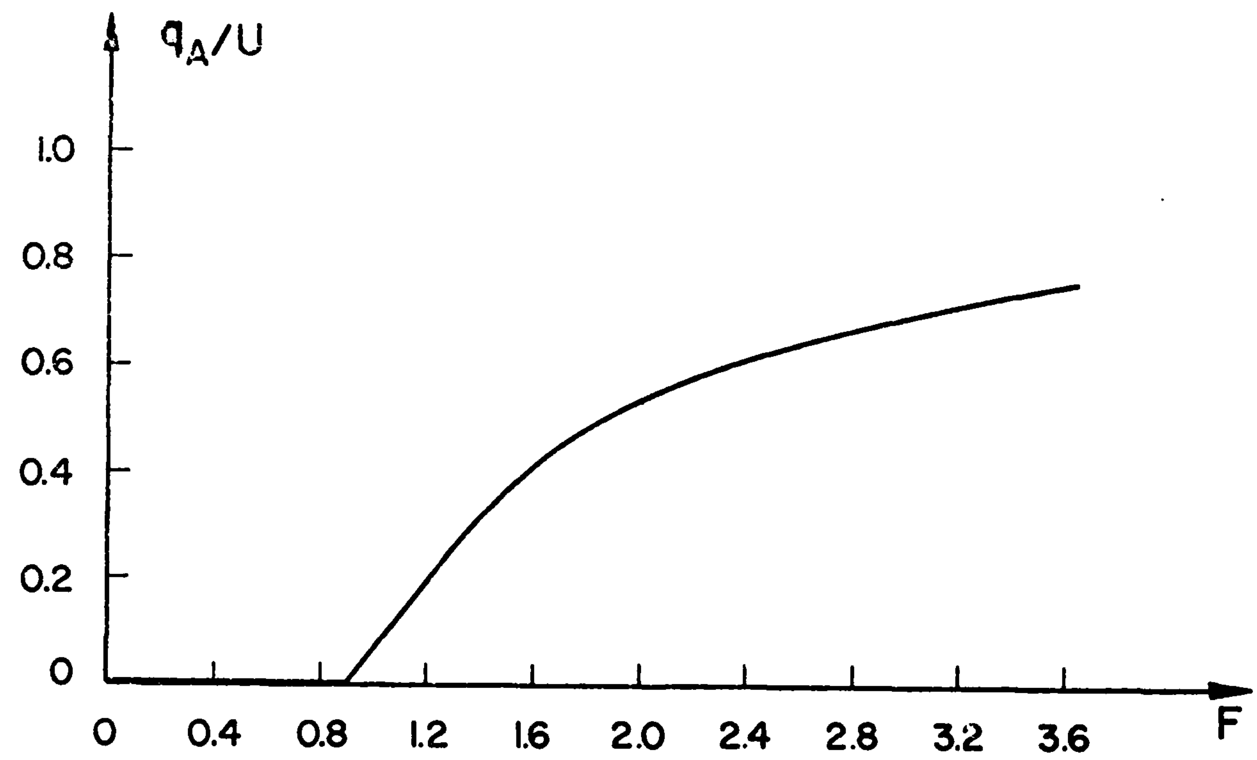

Figure 4: Values of the dimensionless velocity q_A/U at the apex A of the bubble versus F for $\alpha = \infty$.

$$U = 0.5(g\ R)^{1/2}\ . \tag{25}$$

In dimensionless variables (25) becomes

$$R = 4F^2\ . \tag{26}$$

Collins found that (25) was in good agreement with his experimental data. In addition he measured the angle subtended at the center of the arc of circle and found the approximate value 105°. Therefore the width D of the bubble is related to the radius R by the simple formula

$$D = 2\ \sin 52.5°\ R = 1.6R\ . \tag{27}$$

In dimensionless variables, relations (25) and (27) imply the existence of a unique flow characterized by

$$F = F_e \sim 0.4\ . \tag{28}$$

Our numerical results without surface tension do not agree with Collins' experimental data: we found a solution for all values of $F < F_c$ whereas Collins found a unique solution for $F = F_e$. In the next section we show that this discrepancy between theory and experiments is removed by taking into account the effect of surface tension.

5. Solutions with Surface Tension

We used the numerical scheme of Section 3 to compute solutions of the modified problem for various values of F and α.

In Fig. 5 we present values of γ vs F for $\alpha = 40$. As F tends to infinity, γ approaches zero. As F approaches zero, γ oscillates often around $\frac{\pi}{2}$. Fig. 5 suggests that there exists a countably infinite number of values of F for which $\gamma = \frac{\pi}{2}$. The solutions corresponding to these values of F are solutions of the original problem. As α increases, the amplitudes and the wavelengths of the oscillations in Fig. 5 decrease. These results are similar to those obtained in Ref. 3 for the flow in a Hele Shaw cell and in Ref. 4 for the flow past a bubble in a tube. By analogy with those previous works we

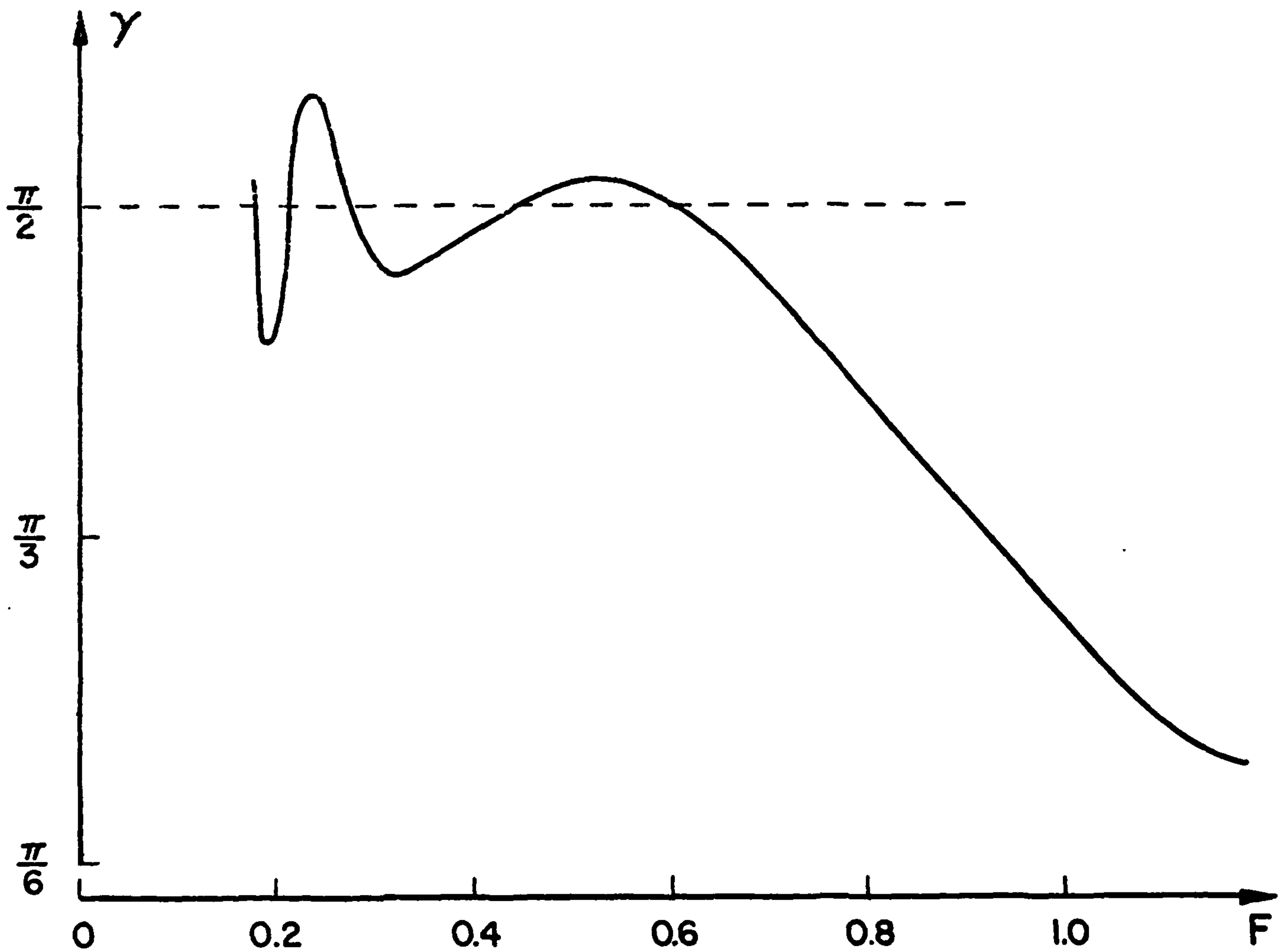

Figure 5: Values of γ versus F for $\alpha =$. The broken line corresponds to the solution defined by (26).

expect that the discrete set of solutions for which $\gamma = \frac{\pi}{2}$ will reduce to a unique solution as $\alpha^{-1} \to 0$. We shall denote by F^* the corresponding value of the Froude number.

In Ref. 4, the value of F^* was obtained by following solutions with $\gamma = \frac{\pi}{2}$ as $\alpha^{-1} \to 0$ (see Fig. 5 in Ref. 4). A similar method was used in Ref. 3.

In the present paper, we shall use a different approach to determine the value of F^*. We first recall that there exists a branch of solutions characterized by $0 < F < F_c$, $\gamma = \pi/2$ and $\alpha^{-1} = 0$ (see Section 4). On the other hand we expect to obtain a unique solution characterized by $F = F^*$ and $\gamma = \pi/2$ as $\alpha^{-1} \to 0$. Keller[9] pointed out that these two facts imply that $F = F^*$ is a bifurcation point on the branch of $0 < F < F_c$, $\gamma = \pi/2$, $\alpha^{-1} = 0$.

We can therefore find the value of F^* by seeking numerically bifurcation points on the branch of solutions $0 < F < F_c$, $\gamma = \pi/2$, $\alpha^{-1} = 0$. For this purpose we use a variant of the scheme of Section 3 in which the problem is reduced to a system of $N + 1$ algebraic equations for the $N + 1$ unknowns b, $\alpha^{-1} A_n$, $n = 1,\dots,N - 1$ to be solved for $\gamma = \frac{\pi}{2}$ and a given value of F. We used this scheme to compute the branch of solutions $0 < F < F_c$, $\gamma = \pi/2$, $\alpha^{-1} = 0$. The numerical results were found to agree with those of Section 4. In particular $\alpha^{-1} \sim 0$ for N large (for example $|\alpha^{-1}| < 10^{-3}$ for $N = 40$ and $|\alpha^{-1}| < 10^{-4}$ for $N = 60$).

For each value of F, we computed the value of the determinant of the Jacobian matrix. We found that it vanishes for $F \sim 0.51$. Therefore $F \sim 0.51$ is a bifurcation point on the branch of solutions $0 < F < F_c$ (Keller[10]) and $F^* \sim 0.51$.

The profile of the bubble corresponding to $F = F^*$ and $\alpha = \infty$ is shown in Fig. 6. The broken line corresponds to a circular arc of radius $R = 4(F^*)^2$.

Fig. 6 shows that our theoretical profile is in very good agreement with the experimental result (26). However, the theoretical value $F^* \sim 0.51$ is larger than the experimental value $F_e \sim 0.4$. This is presumably due to our simplified model of the wake and to the difficulty of realizing truly two dimensional flows experimentally.

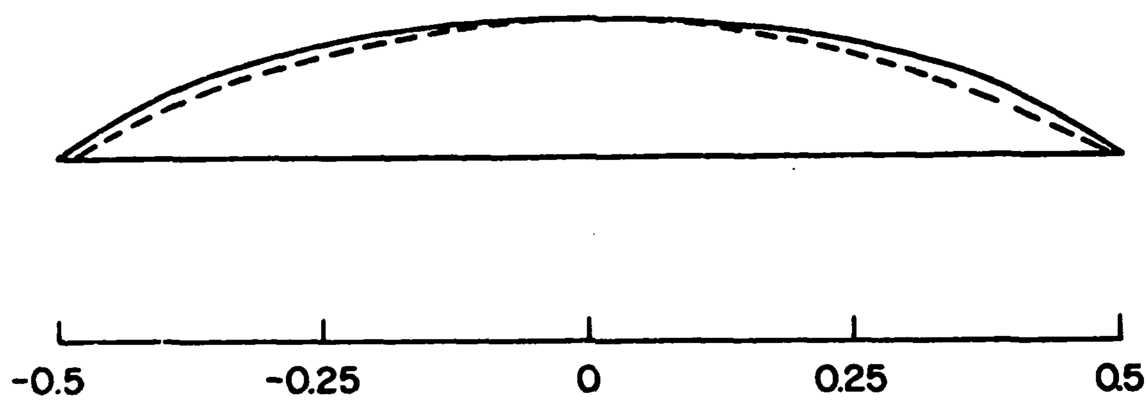

Figure 6: Bubble profile for $F = F^* \sim 0.51$. The broken line corresponds to an arc of circle of radius $4F^*$.

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The motion of a two-dimensional bubble rising at a constant velocity in an unbounded fluid is solved by series truncation. It is assumed there is a wake of stagnant liquid extending to infinity below the bubble. Both the effects of gravity g and surface tension T are taken into account. It is shown that the problem is characterized by a continuum of solutions when $T = 0$ and by a discrete set of solutions when $T > 0$. In addition, a unique solution is obtained in the limit as the surface tension approaches zero. The corresponding profile of the bubble is found to be in good agreement with experimental data.

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